The Structure Matrix and a Generalization of Ryser's Maximum Term Rank Formula

T. S. Michael*
Mathematics Department
United States Naval Academy
Annapolis, Maryland 21402

Dedicated to the memory of Scott R. Johnson.

Submitted by Richard A. Brualdi

ABSTRACT

Let $\mathfrak{A}(R, S)$ denote the class of all (0,1)-matrices with row sum vector $R = (r_1, r_2, \dots, r_m)$ and column sum vector $S = (s_1, s_2, \dots, s_n)$. Suppose that $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_m$ and $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$. A theorem of Ford and Fulkerson asserts that the class $\mathfrak{A}(R, S)$ is nonempty if and only if $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$ and the structure matrix $T(R, S)$ associated with R and S is nonnegative. We show that this theorem remains valid under a weaker hypothesis on R and S than monotonicity. This stronger result enables us to prove a generalization of Ryser's maximum term rank formula. Moreover, if we extend our definitions suitably, then our results are valid for classes of matrices with elements in the set $\{0, 1, \dots, q\}$ in place of $\{0, 1\}$. We work in the more general context throughout. Our theorems give information concerning the existence and size of matchings and other nearly regular subgraphs in bipartite graphs with prescribed degree sequences.

1. AN OVERVIEW

Throughout this paper q denotes a positive integer, and

$$R = (r_1, r_2, \dots, r_m) \quad \text{and} \quad S = (s_1, s_2, \dots, s_n)$$

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denote sequences of nonnegative integers. We let

$$\mathfrak{A}_q(R,S)$$

denote the class of all matrices $A=[a_{ij}]$ of size m by n such that

$$a_{ij}\in\{0,1,\dots,q\} \qquad \text{for} \quad i=1,2,\dots,m \text{ and } j=1,2,\dots,n, \tag{1.1}$$

$$\sum_{j=1}^{n} a_{ij}=r_i \qquad \text{for} \quad i=1,2,\dots,m, \tag{1.2}$$

$$\sum_{i=1}^{m} a_{ij}=s_j \qquad \text{for} \quad j=1,2,\dots,n. \tag{1.3}$$

Thus each matrix in the class $\mathfrak{A}_q(R,S)$ is a $(0,1,\dots,q)$-*matrix* with *row sum vector* R and *column sum vector* S. The *structure matrix*

$$T=T_q(R,S)=\left[t_{ij}\right] \qquad (i=0,1,\dots,m;\quad j=0,1,\dots,n)$$

(of *multiplicity* q) of R and S has as its entries the $(m+1)(n+1)$ *structure constants*

$$t_{ij}=t_{ij}(R,S,q)=qij+\sum_{k>i}r_k-\sum_{k\leqslant j}s_k, \tag{1.4}$$

where $i=0,1,\dots,m$ and $j=0,1,\dots,n$. (Empty summations are assigned the value 0.)

The vector $V=(v_1,v_2,\dots,v_p)$ is *monotone* provided

$$v_j-v_i\leqslant 0 \qquad \text{for} \quad 1\leqslant i<j\leqslant p.$$

The vector V is *nearly monotone* provided

$$v_j-v_i\leqslant 1 \qquad \text{for} \quad 1\leqslant i<j\leqslant p.$$

The class $\mathfrak{A}_q(R,S)$ is (*nearly*) *monotone* provided both R and S are (nearly) monotone. The rows and columns of an integral matrix may always be permuted so that its row sum and column sum vectors become (nearly) monotone. Our first result extends a fundamental theorem of Ford and Fulkerson [4, pp. 79–82].

THEOREM 1.1. *The nearly monotone class* $\mathfrak{A}_q(R, S)$ *is nonempty if and only if* $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$ *and the structure matrix* $T_q(R, S)$ *is nonnegative.*

The Ford-Fulkerson theorem deals with the monotone class $\mathfrak{A}_1(R, S)$. Their stipulation that R and S be monotone is a natural one; the components of potential row sum and column sum vectors may usually be permuted. But the stronger result of Theorem 1.1 streamlines our proof of the following generalization of Ryser's maximum term rank formula.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be nonnegative integral matrices of size m by n. We write

$$B \leqslant A \quad \text{provided} \quad b_{ij} \leqslant a_{ij}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The *weight* of B, denoted by $\tau(B)$, is the sum of all of the entries of B. The matrix B is *nearly biregular* provided its row sums differ by at most 1 and its column sums differ by at most 1.

THEOREM 1.2. *Let* $T = T_q(R, S) = [t_{ij}]$ *denote the structure matrix of the nonempty, monotone class* $\mathfrak{A}_q(R, S)$. *Let* τ *be a nonnegative integer, and suppose*

$$\tau = mh + a = nk + b, \tag{1.5}$$

where h, k, a, *and* b *are integers with* $0 \leqslant a \leqslant m$ *and* $0 \leqslant b \leqslant n$. *Then there exist a matrix* A *in* $\mathfrak{A}_q(R, S)$ *and a nearly biregular matrix* B *of weight* τ *with* $B \leqslant A$ *if and only if the inequality*

$$\tau \leqslant t_{ij} + \min\{i, a\} + \min\{j, b\} + hi + kj \tag{1.6}$$

holds for $i = 0, 1, \ldots\ m$ *and* $j = 0, 1, \ldots, n$.

If $q = 1$ and $\tau \leqslant \min\{m, n\}$, then we may select $h = k = 0$ in (1.5), and Ryser's maximum term rank formula [7; 8, p. 75] follows from (1.6). (See Corollary 5.1 below.)

Our proof of Theorem 1.2 begins along the same lines as the first part of the proof of Ryser's formula given by Brualdi and Ross [3]. Theorem 1.1 completes the proof in the following manner. In Theorem 1.2 the matrix $A - B$ is in a class $\mathfrak{A}_q(\hat{R}, \hat{S})$, where $\hat{R}$ and $\hat{S}$ are nearly monotone vectors obtained by decreasing appropriate components of R and S. Theorem 1.1 applies, and we show that the inequalities in (1.6) hold if and only if the structure matrix $T_q(\hat{R}, \hat{S})$ is nonnegative.

Each matrix in the class $\mathfrak{A}_q(R, S)$ corresponds in the usual way to a bipartite q-multigraph with degree sequences R and S. Thus the results in this paper may be reformulated as results about classes of bipartite multigraphs with prescribed degrees. If $m = n$, then each matrix in the class $\mathfrak{A}_q(R,S)$ corresponds in the usual way to a directed q-multigraph with outdegree sequence R and indegree sequence S. Thus, in this case our results may also be recast as results about classes of directed multigraphs with prescribed degrees.

2. SOME ELEMENTARY PROPERTIES OF THE STRUCTURE MATRIX

In our first lemma we list several properties of the structure constants

$$t_{ij} = qij + \sum_{k>i} r_k - \sum_{k\leqslant j} s_k \tag{2.1}$$

and the structure matrix $T_q(R, S) = [t_{ij}]$. We emphasize that no monotonicity assumptions are imposed on R and S.

LEMMA 2.1. *Suppose that the vectors R and S satisfy*

$$r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n. \tag{2.2}$$

Then

$$t_{ij} = qij - \sum_{k\leqslant i} r_k + \sum_{k>j} s_k. \tag{2.3}$$

The entries in row i of T satisfy the recurrence relation

$$t_{ij} - t_{i,j-1} = qi - s_j \qquad (j = 1, 2, \ldots, n). \tag{2.4}$$

The entries in column j of T satisfy the recurrence relation

$$t_{ij} - t_{i-1,j} = qj - r_i \qquad (i = 1, 2, \ldots, m). \tag{2.5}$$

Moreover, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ the recurrence relation

$$\left(t_{ij} + t_{i-1,j-1}\right) - \left(t_{i-1,j} + t_{i,j-1}\right) = q \tag{2.6}$$

also holds. The entries in row 0 *and column* 0 *of* T *are*

$$\begin{aligned}
t_{0n} &= 0, & t_{m0} &= 0,\\
t_{0,n-1} &= s_n, & t_{m-1,0} &= r_m,\\
t_{0,n-2} &= s_n + s_{n-1}, & t_{m-2,0} &= r_m + r_{m-1},\\
&\vdots & &\vdots\\
t_{00} &= s_n + s_{n-1} + \cdots + s_1, & t_{00} &= r_m + r_{m-1} + \cdots + r_1.
\end{aligned} \tag{2.7}$$

Equations (2.3)–(2.7) are immediate consequences of the definition (2.1) and Equation (2.2). We omit the proofs. The above relationships and many other properties of structure matrices are found in the survey article by Brualdi [2] for the special case $q = 1$. We remark that (2.7) and the recurrence relation (2.6) facilitate the computation of the $(m+1)(n+1)$ entries of the structure matrix T.

Our next result motivates our definition (2.1) of the structure constants. The proof uses the usual counting arguments for results of this type.

LEMMA 2.2. *Suppose that the class* $\mathfrak{A}_q(R, S)$ *is nonempty. Then*

$$r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n,$$

and the structure matrix $T_q(R, S) = [t_{ij}]$ *is nonnegative.*

Proof. Suppose that $A \in \mathfrak{A}_q(R, S)$. Then

$$r_1 + r_2 + \cdots + r_m = \tau(A) = s_1 + s_2 + \cdots + s_n.$$

For $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$ consider the decomposition

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix},$$

where the submatrices W and Z are of sizes i by j and $m - i$ by $n - j$, respectively. (If $i = 0$ or m or if $j = 0$ or n, then some of the submatrices are vacuous.) Let J_{mn} denote the matrix of 1's of size m by n. Consider the decomposition

$$\bar{A} = qJ_{m,n} - A = \begin{bmatrix} \overline{W} & \overline{X} \\ \overline{Y} & \overline{Z} \end{bmatrix},$$

where $\overline{W}$ and $\overline{Z}$ are of sizes i by j and $m-i$ by $n-j$, respectively. The matrices A and $\bar{A}$ are both nonnegative. Hence

$$\begin{aligned} 0 &\leqslant \tau(Z) + \tau(\overline{W}) = qij + \tau(Z) - \tau(W) \\ &= qij + \tau(Z) + \tau(Y) - \tau(Y) - \tau(W) \\ &= qij + \sum_{k>i} r_k - \sum_{k\leqslant j} s_k = t_{ij}. \end{aligned}$$ ■

3. PROOF OF THEOREM 1.1

One implication follows from Lemma 2.2. Now suppose that $r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$ and that T is nonnegative. We induct on the parameter $\tau = r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n$. Suppose that $\tau = 0$. Then $R = (0, 0, \ldots, 0)$ and $S = (0, 0, \ldots, 0)$. The structure matrix $T = [qij]$ is nonnegative, and $\mathfrak{A}_q(R, S)$ consists of a matrix of 0's.

We henceforth suppose that $\tau > 0$. Without loss of generality $r_i > 0$ for $i = 1, 2, \ldots, m$ and $s_j > 0$ for $j = 1, 2, \ldots, n$. We define the index

$$e = \min\{i : r_i \geqslant r_k \text{ for } k = 1, 2, \ldots, m\}.$$

Because R is nearly monotone,

$$1 \leqslant i < e \quad \text{implies} \quad r_i = r_e - 1. \tag{3.1}$$

Let E_k denote a unit vector with a 1 in position k and a 0 in each other position. We define the vectors

$$\hat{R} = R - E_e \quad \text{and} \quad \hat{S} = S - E_n.$$

Then $\hat{R}$ and $\hat{S}$ are nearly monotone. Also, the sum of the components of each of $\hat{R}$ and $\hat{S}$ equals $\tau - 1$. The entries of the structure matrix

$$\hat{T} = T_q(\hat{R}, \hat{S}) = [\hat{t}_{ij}]$$

are given by

$$\hat{t}_{ij} = \begin{cases} t_{ij} - 1 & \text{if } 0 \leqslant i < e \text{ and } 0 \leqslant j < n, \\ t_{ij} & \text{otherwise.} \end{cases} \tag{3.2}$$

We assert that $\hat{T}$ is nonnegative. For suppose that $0 \leqslant i < e$ and $0 \leqslant j < n$. If $r_e \geqslant qj + 1$, then by (3.1) we have

$$\begin{aligned} t_{ij} &= qij + \sum_{k>i} r_k - \sum_{k\leqslant j} s_k \\ &= qej + \sum_{k>e} r_k - \sum_{k\leqslant j} s_k - q(e-i)j + (e-i)(r_e - 1) + 1 \\ &= t_{ej} + (e-i)(r_e - qj - 1) + 1 \geqslant 1. \end{aligned}$$

On the other hand, if $r_e < qj + 1$, then by (2.3) and (3.1) we have

$$\begin{aligned} t_{ij} &= qij - \sum_{k\leqslant i} r_k + \sum_{k>j} s_k = qij - i(r_e - 1) + \sum_{k>j} s_k \\ &= i(qj + 1 - r_e) + \sum_{k>j} s_k \geqslant s_n \geqslant 1. \end{aligned}$$

Thus the nonnegativity of $\hat{T}$ follows from (3.2) and the nonnegativity of T.

By induction there is a matrix $\hat{A} = [\hat{a}_{ij}]$ in the class $\mathfrak{A}_q(\hat{R}, \hat{S})$. Let E_{ij} denote the matrix of size m by n with a 1 in the (i, j) position and a 0 in each other position. If $\hat{a}_{en} < q$, then $\hat{A} + E_{en} \in \mathfrak{A}_q(R, S)$. Suppose that $\hat{a}_{en} = q$. The inequality $0 \leqslant t_{en} = e[qn - (r_e - 1)] - 1$ implies $r_e - 1 < qn$. Thus $\hat{a}_{ek} < q$ for some index k. By the minimality of $s_n - 1$ there is an index h such that $\hat{a}_{hn} < \hat{a}_{hk}$. The matrix $\hat{A} - E_{hk} + E_{ek} + E_{hn} \in \mathfrak{A}_q(R, S)$. ■

4. PROOF OF THEOREM 1.2

LEMMA 4.1. *Suppose that the vectors* $R' = (r'_1, r'_2, \ldots, r'_m)$ *and* $S' =$

$(s_1', s_2', \ldots, s_n')$ *satisfy*

$$r_i - r_i' \in \{h, h+1\} \qquad \text{for} \quad i = 1, 2, \ldots, m,$$

$$s_j - s_j' \in \{k, k+1\} \qquad \text{for} \quad j = 1, 2, \ldots, n$$

for some nonnegative integers h and k. Then there exist matrices A in $\mathfrak{A}_q(R, S)$ and A' in $\mathfrak{A}_q(R', S')$ with $A' \leqslant A$ if and only if both of the classes $\mathfrak{A}_q(R, S)$ and $\mathfrak{A}_q(R', S')$ are nonempty.

Brualdi and Ross [3] have given a proof of Lemma 4.1 in the special case $q = 1$ and $h = k = 0$ based on the ideas in the proof by Lovász [5] of the 1-factor theorem. Anstee has observed that essentially the same argument establishes Lemma 4.1 for $q = 1$. (See [1, Corollary 3.5] and [2, Theorem 10.8].) The generalization to $q \geqslant 1$ presents no new difficulties. We omit the proof.

LEMMA 4.2. *Suppose that $B \leqslant A$, where B is a nearly biregular matrix of weight τ and A is in the monotone class $\mathfrak{A}_q(R, S)$. Then $B' \leqslant A'$ for some nearly biregular matrix B' of weight τ with monotone row sum and column sum vectors and some matrix A' in $\mathfrak{A}_q(R, S)$.*

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ satisfy the hypothesis of the lemma. Suppose that the sum of row $e+1$ of B exceeds the sum of row e by 1. Then $b_{ef} < b_{e+1,f}$ for some index f. If $a_{ef} > b_{ef}$, then we define $\hat{A} = A$ and $\hat{B} = B - E_{e+1,f} + E_{ef}$. We henceforth suppose that $a_{ef} = b_{ef}$. By the monotonicity of R there is an index $j \neq f$ such that $a_{ej} - b_{ej} > a_{e+1,j} - b_{e+1,j}$. If $b_{e+1,j} > 0$, then we define $\hat{A} = A$ and $\hat{B} = B + E_{ej} - E_{e+1,j}$. Now suppose that $b_{e+1,j} = 0$. In this final case we define $\hat{B} = B - E_{e+1,f} + E_{ef}$ and $\hat{A} = A - E_{ej} - E_{e+1,f} + E_{ef} + E_{e+1,j}$. In all cases the matrices $\hat{A}$ and $\hat{B}$ satisfy the hypothesis of the lemma, but the sum of row e of $\hat{B}$ now exceeds the sum of row $e+1$ by 1. Iteration of this process shows that for some matrix A'' in $\mathfrak{A}_q(R, S)$ there is a nearly biregular matrix B'' of weight τ with monotone row sum vector and with $B'' \leqslant A''$. The column sum vector of B'' is the same as that of B. A similar argument on the columns of B'' completes the proof. ∎

We now prove Theorem 1.2. By Lemma 4.1 and Lemma 4.2 there exist mattrices A and B that satisfy the conditions of Theorem 1.2 if and only if the class $\mathfrak{A}_q(\hat{R}, \hat{S})$ is nonempty, where

$$\hat{R} = (\hat{r}_1, \hat{r}_2, \dots, \hat{r}_m) = R - (\underbrace{h+1, \dots, h+1}_{a}, \underbrace{h, \dots, h}_{m-a}),$$

$$\hat{S} = (\hat{s}_1, \hat{s}_2, \dots, \hat{s}_n) = S - (\underbrace{k+1, \dots, k+1}_{b}, \underbrace{k, \dots, k}_{n-b}).$$

The vectors $\hat{R}$ and $\hat{S}$ are nearly monotone. By Theorem 1.1 the class $\mathfrak{A}_q(\hat{R}, \hat{S})$ is nonempty if and only if the structure matrix $T_q(\hat{R}, \hat{S}) = [\hat{t}_{ij}]$ is nonnegative. Now for $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$ we have

$$\begin{aligned}
\hat{t}_{ij} &= q \cdot ij + \sum_{p>i} \hat{r}_p - \sum_{p \leqslant j} \hat{s}_p \\
&= q \cdot ij + \sum_{p>i} (r_p - h) - \max\{0, a - i\} - \sum_{p \leqslant j} (s_p - k) + \min\{j, b\} \\
&= q \cdot ij + \sum_{p>i} r_p - \sum_{p \leqslant j} s_p + \min\{i, a\} - a + \min\{j, b\} - h(m - i) + kj \\
&= t_{ij} - \tau + \min\{i, a\} + \min\{j, b\} + hi + kj.
\end{aligned}$$

Therefore $T_q(\hat{R}, \hat{S})$ is nonnegative if and only if (1.6) holds. ■

5. CONSEQUENCES OF THEOREM 1.2

Let A be a (0, 1)-matrix. A set of 1's in A is *independent* provided no two of the 1's are in the same row or column of A. The *term rank* of A, denoted by $\rho(A)$, is the maximum cardinality of an independent set of 1's in A. Thus $\rho(A)$ equals the maximum cardinality of a matching in the bipartite graph associated with A. Suppose that the class $\mathfrak{A}_1(R, S)$ is nonempty. We define

$$\bar{\rho} = \bar{\rho}(R, S) = \max\{\rho(A) : A \in \mathfrak{A}_1(R, S)\}.$$

Ryser deduced the following formula for the maximum term rank $\bar{\rho}$ from a decomposition theorem [7; 8, p. 75]. Brualdi and Ross [3] provided a more natural approach. Our search for an even simpler proof led to the discovery of Theorem 1.1 and Theorem 1.2.

COROLLARY 5.1 (Ryser). *Let $T = T_1(R, S) = [t_{ij}]$ be the structure matrix of the nonempty, monotone class $\mathfrak{A}_1(R, S)$. Then*

$$\bar{\rho} = \min_{i,j} \{t_{ij} + i + j\}, \tag{5.1}$$

where the minimum extends over all $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$.

Proof. Let ρ be an integer with $0 \leqslant \rho \leqslant \min\{m, n\}$. There exists a matrix in $\mathfrak{A}_1(R, S)$ with a set of ρ independent 1's if and only if for some matrix A in $\mathfrak{A}_1(R, S)$ there is a nearly biregular matrix B of weight ρ with $B \leqslant A$. In Theorem 1.2 we select $q = 1$, $h = k = 0$, and $\tau = a = b = \rho$. By (1.6) there exists a matrix in $\mathfrak{A}_1(R, S)$ with term rank ρ if and only if the inequality

$$\rho \leqslant t_{ij} + \min\{i, \rho\} + \min\{j, \rho\} \tag{5.2}$$

holds for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. If $i \geqslant \rho$ or if $j \geqslant \rho$, then (5.2) clearly holds. Suppose that $i < \rho$ and $j < \rho$. Then (5.2) holds if and only if $\rho \leqslant t_{ij} + i + j$. This proves (5.1). ∎

We conclude with another corollary of Theorem 1.2.

COROLLARY 5.2. *Let $T = T_q(R, S) = [t_{ij}]$ be the structure matrix of the nonempty, monotone class $\mathfrak{A}_q(R, S)$. Suppose that h and k are nonnegative integers with*

$$hm = kn = \tau.$$

Let

$$H = \underbrace{(h, h, \ldots, h)}_{m} \quad \textit{and} \quad K = \underbrace{(k, k, \ldots, k)}_{n}.$$

Then $B \leqslant A$ holds for some matrices A in $\mathfrak{A}_q(R, S)$ and B in $\mathfrak{A}_q(H, K)$ if and only if the inequality

$$\tau \leqslant \left(\frac{mn}{mn - (ni + mj)} \right) t_{ij} \tag{5.3}$$

holds for all i and j for which $mn > ni + mj$.

Proof. In Theorem 1.2 we select $a = b = 0$. By (1.6) our matrices A and B exist if and only if the inequality

$$\tau \leqslant t_{ij} + hi + kj = t_{ij} + \tau\left(\frac{ni + mj}{mn}\right) \tag{5.4}$$

holds for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. If $mn \leqslant ni + mj$, then (5.4) clearly holds. Suppose that $mn > ni + mj$. Then (5.4) holds if and only if (5.3) holds. ∎

We remark that it is possible to associate a structure matrix with the class of all r-multigraphs with a prescribed degree sequence. Each result in this paper possesses a valid analogue in the new setting [6]. For instance, the counterpart of Ryser's maximum term rank formula is a formula for the maximum cardinality of a matching among all graphs with a prescribed degree sequence.

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